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ORIGINAL RESEARCH

Existence and Hyers–Ulam stability results for nonlinear fractional systems with coupled nonlocal initial conditions

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Abstract In this paper, we prove the existence of solutions for two kinds of nonlinear fractional differential systems with coupled nonlocal initial conditions, the approach is based on the fixed point theorem of Perov and the choice of suitable norm for vectors. Further, Hyers–Ulam stability problems are discussed for these two kinds of nonlinear fractional systems. Two examples are presented to illustrate the theory.

Keywords Nonlinear fractional differential equations · Nonlocal initial conditions · Hyers–Ulam stability · Vector-valued norm

Mathematics Subject Classification 26A33 · 45N05

1 Introduction

Fractional differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. There has been significant development in fractional differential equations in recent years; One can see the monographs of Podlubny [1], V. Lakshmikantham et al. [2] Miller and Ross [3], Kilbas [4] and the research papers of Agarwal [5,6], Ahmad and Nieto [7], Deng et al. [8], Bai [9], Wang et al. [10], Zhou et al. [11] and Zhao [12]. In these previous works, Chauchy problems, nonlocal

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problems, impulsive problems, stability problems and boundary value problems of all kinds of fractional differential equations are discussed.

Nonlocal problems occurred naturally when modelling physical problems, Nonlocal problems (such as: nonlocal initial problems [12,14,16], nonlocal boundary problems [13,15]) for several classes of differential equations and systems were extensively discussed in literature by various methods. This paper is motivated by the works [12,17,18] in which systems with coupled nonlocal conditions and Hyers–Ulams stability of differential equations have been discussed, we intend to study and develop some existence and Hyers–Ulam stability results given in [11,19–23].

In this paper, we discuss two kinds of fractional order nonlinear differential systems as following

$$\begin{cases} D_0^\alpha u_1(t) = f_1\big(t, u_1(t), u_2(t), \dots, u_n(t)\big), \\ D_0^\alpha u_2(t) = f_2\big(t, u_1(t), u_2(t), \dots, u_n(t)\big), \\ \cdots \\ D_0^\alpha u_n(t) = f_n\big(t, u_1(t), u_2(t), \dots, u_n(t)\big), \end{cases} \tag{1.1}$$

for a.e.$t \in [0, 1]$, subject to the coupled nonlocal conditions

$$\begin{cases} u_1(0) = \nu_{11}[u_1] + \nu_{12}[u_2] + \cdots + \nu_{1n}[u_n], \\ u_2(0) = \nu_{21}[u_1] + \nu_{22}[u_2] + \cdots + \nu_{2n}[u_n], \\ \cdots \\ u_n(0) = \nu_{n1}[u_1] + \nu_{n2}[u_2] + \cdots + \nu_{nn}[u_n]. \end{cases} \tag{1.2}$$

And

$$\begin{cases} D_0^\alpha (D+\lambda) u_1(t) = f_1\big(t, u_1(t), u_2(t), \dots, u_n(t)\big), \\ D_0^\alpha (D+\lambda) u_2(t) = f_2\big(t, u_1(t), u_2(t), \dots, u_n(t)\big), \\ \cdots \\ D_0^\alpha (D+\lambda) u_n(t) = f_n\big(t, u_1(t), u_2(t), \dots, u_n(t)\big), \end{cases} \tag{1.3}$$

for a.e.$t \in [0, 1]$, subject to the coupled nonlocal conditions

$$\begin{cases} u_1(0) = 0, u_2(0) = 0, \dots, u_n(0) = 0 \\ u_1'(0) = \nu_{11}[u_1] + \nu_{12}[u_2] + \cdots + \nu_{1n}[u_n], \\ u_2'(0) = \nu_{21}[u_1] + \nu_{22}[u_2] + \cdots + \nu_{2n}[u_n], \\ \cdots \\ u_n'(0) = \nu_{n1}[u_1] + \nu_{n2}[u_2] + \cdots + \nu_{nn}[u_n]. \end{cases} \tag{1.4}$$

respectively. Here, D_0^α is Caputo fractional derivative of order $0 < \alpha \leq 1$, $f_i : [0, 1] \times R^n \to R (i = 1, 2, \dots, n)$ are L_1-Carathèodary functions,$\nu_{ij} : C[0, 1] \to R, i, j = 1, 2, \dots, n$ are linear and continuous functions and $\lambda > 0, \lambda \in R$ is a constant.

2 Preliminaries

We need some basic definitions,notations and theorems that shall be used in remainder of the paper.Let $X \subset R^n$ be a Banach space and $R_+ = [0, +\infty)$.

Definition 2.1 ([1,4]) The fractional integral operator of order $\alpha > 0$ of function $f \in L(R_+)$ is defined as

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 ([1]) The Caputo fractional derivative order $\alpha > 0, n-1 < \alpha \leq n$ is defined as

$$^C D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $n-1$.

Properties 2.3 ([4]) *For* $\alpha, \beta > 0$ *and* f *a suitable function we have*

$$\begin{aligned} &(i) \quad I_a^\alpha I_a^\beta f(t) = I_a^{\alpha+\beta} f(t); \\ &(ii) \quad I_a^{\alpha C} D_a^\alpha f(t) = f(t) - f(a), \quad 0 < \alpha \leq 1; \\ &(iii) \ ^C D_a^{\alpha C} D_a^\beta f(t) \neq {}^C D_a^{\alpha+\beta} f(t). \end{aligned}$$

Definition 2.4 ([7]) Let X be a nonempty set. A vector-valued metric on X is a map $d : X \times X \to R, n \in N$ with the following properties

(i) $d(u,v) \geq 0, \quad \forall u,\ v \in X;\ d(u,v) = 0$ if and only if $u = v$.
(ii) $d(u,v) = d(v,u), \quad \forall u,\ v \in X.$
(iii) $d(u,v) \leq d(v,w) + d(w,v), \quad \forall u,\ v,\ w \in X.$

The pair (X,d) is called a generalized metric space for such a space convergence and completeness are similar to those in usual metric space.

Definition 2.5 ([11]) Let (X,d) be generalized metric space, the map $T : X \to X$ is called a contraction if there exists a convergnet to zero matrix M such that $d(T(u), T(v)) \leq M d(u,v), u, v \in X$. In this case, M is called $T's$ Lipschitz matrix.

Theorem 2.6 ([11] Perov) *Let* M *be a square matrix with nonnegative elements. The following are equivalent*

(a) M *is convergent to zero.*
(b) *the eigenvalues of* M *are located inside the unix disc of complex plane.*
(c) $I - M$ *is nonsingular and* $(I-M)^{-1}$ *has nonnegative elements.*

Theorem 2.7 ([11]) *Let* (X,d) *be a complete generalized metric space* $T : X \to X$ *a contractive map with a Lipschitz matrix* M. *Then* T *has a unique fixed point* u^* *and each* $u_0 \in X$, *one has*

$$d\big(T^k(u_0), u^*\big) \leq M^k (I-M)^{-1} d\big(u_0, T(u_0)\big), \quad \forall k \in N.$$

Theorem 2.8 ([14] Hölder inequality) *Assume that* $\sigma, p \geq 1$ *and* $\frac{1}{\sigma} + \frac{1}{p} = 1$, *if* $l, m \in L^p(J, R)$, *then for* $1 \leq p < \infty$, $lm \in L^1(J, R)$ *and*

$$\|lm\|_{L^1 J} = \|l\|_{L^\sigma J} \|m\|_{L^p J}. \tag{2.3}$$

Let $C(J, X)$ *be the Banach space of continuous functions* $x(t)$ *with* $x \in X$ *for* $t \in J = [0, 1]$ *and*

$$\|x\|_{C(J,X)} = (\|x_1\|, \|x_2\|, \ldots, \|x_n\|)^T$$

where $\|x_i\|_{C(J,X)} = \max_{t \in J} |x_i(t)|, i = 1, 2, \ldots, n$. *We denote by* $\mathsf{M}n \times n$ *the sets of all square matrices of order n, and* $C[J, \mathsf{M}_{n\times n}]$ *the Banach space with continuous elements* $m_{ij}(t) : J \to R, 1 \leq i, j \leq n$, *and for* $M_{n\times n} \in C[J, \mathsf{M}_{n\times n}]$ *with the norm*

$$\|M\|_{C(J,X)} = (\|m_{ij}\|)_{1 \leq i,j \leq n}, \tag{2.4}$$

where $\|m_{ij}\| = \max_{t \in J}(|m_{ij}(t)|)$.

3 Existence and Hyers–Ulam stability results of system (1.1)–(1.2)

The problems (1.1), (1.2) can be rewritten in the vector form

$$\begin{cases} D_0^\alpha u(t) = F(t, u(t)), \\ u(0) \quad\;\; = \nu[u]. \end{cases} \tag{3.1}$$

where $u(t) = \big(u_1(t), u_2(t), \ldots, u_n(t)\big)^T$, $F\big(t, u(t)\big) = \big(f_1\big(t, u(t)\big), f_2\big(t, u(t)\big), \ldots, f_n\big(t, u(t)\big)\big)^T$, and $\nu[u] = (\nu_1[u], \nu_2[u], \ldots, \nu_n[u])^T$, here $\nu_i[u] = \nu_{i1}[u_1] + \nu_{i2}[u_2] + \cdots + \nu_{in}[u_n]$.

Let us integrate the system (3.1) over [0, 1],and obtain

$$u(t) = C + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F\big(s, u(s)\big) ds, \tag{3.2}$$

where $C = (C_1, C_2, \ldots, C_n)^T \in R^n$ is a vector. Secondly, the initial condition $u(0) = \nu[u]$ and linearity of ν give

$$\begin{aligned} C &= \nu\left[C + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F\big(s, u(s)\big) ds\right] \\ &= C\nu[1] + \tfrac{1}{\Gamma(\alpha)} \nu\left[\int_0^t (t-s)^{\alpha-1} F\big(s, u(s)\big) ds\right], \end{aligned} \tag{3.3}$$

and

$$\big(I - \nu[1]\big)C = \frac{1}{\Gamma(\alpha)}\nu\Bigg[\int\limits_0^t (t-s)^{\alpha-1}F\big(s, u(s)\big)ds\Bigg]. \tag{3.4}$$

Then, assuming that matrix $I - \nu[1]$ is nonsingular, where I is the unix of order n, (3.4) gives

$$C = \big(I - \nu[1]\big)^{-1}\frac{1}{\Gamma(\alpha)}\nu\Bigg[\int_0^t (t-s)^{\alpha-1}F\big(s, u(s)\big)ds\Bigg].$$

Therefore, the problem (3.1) is equivalent to the integral type equation

$$u(t) = \big(I - \nu[1]\big)^{-1}\frac{1}{\Gamma(\alpha)}\nu\Bigg[\int_0^t (t-s)^{\alpha-1}F\big(s, u(s)\big)ds\Bigg] + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F\big(s, u(s)\big)ds, \tag{3.5}$$

in the space $C([0, 1], X)$.

We make the following assumptions

(a) The functions $f_i (i = 1, 2, \ldots)$ satisfy Lipschitz conditions of the form

$$\Big|f_i\big(t, u(t)\big) - f_i\big(t, w(t)\big)\Big| \le b_{i1}(t)\big|u_1 - w_1\big| + b_{i2}(t)\big|u_2 - w_2\big| + \cdots + b_{in}(t)\big|u_n - w_n\big|, \quad i=1, 2, \ldots, n, \;\; for\; t \in [0, 1] \tag{3.6}$$

for all $u, w \in X$, and there exists $0 < \alpha_1 < \alpha, \alpha_1 \in R$ such that $b_{ij}(t) \in L_{\alpha_1}((0, 1); R_+), 1 \le i, j \le n$.

Using vector notations we can rewrite the condition (3.6) as follows

$$\big\|f\big(t, u(t)\big) - f\big(t, w(t)\big)\big\| \le B(t)\big\|u - w\big\| \tag{3.7}$$

where $B(t) = \big(b_{ij}(t)\big)_{1\le i,j\le n}$, and $B(t) \in L_{\alpha_1}\big((0, 1); B(t) \in \mathsf{M}_{n\times n}\big)$.

(b) The matrix $\big(I - \nu[1]\big)$ is nonsingular.

Theorem 3.1 *Assume $f_i : [0, 1]\times R^n \to R, (i = 1, 2, \ldots, n)$ are L^1–Carathéodory functions and satisfy the assumption* (a)*, $\nu_{ij} : C[0, 1] \to R, i, j = 1, 2, \ldots, n$ are linear and continuous functions and satisfy the assumption.* (b) *If the spectral radius of the matrix*

$$\frac{1}{\Gamma(\alpha)}\big|\big(I - \nu[1]\big)^{-1}\|\nu\| + I\big|\|B\|_{L_{\frac{1}{\alpha_1}}[0,1]}, \tag{3.8}$$

is less than 1, then the problem (3.1) has a unique solution in X.

Proof The conclusion will follow once we have shown that the operator T defined by

$$Tu = \big(I - \nu[1]\big)^{-1}\frac{1}{\Gamma(\alpha)}\nu\left[\int_0^t (t-s)^{\alpha-1}F\big(s,u(s)\big)ds\right] + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F\big(s,u(s)\big)ds, \tag{3.9}$$

is contractively with respect to a suitable vector norm on $C\big([0,1],X\big)$.

For any functions $u, w \in C\big([0,1],X\big)$, we have

$$\begin{aligned}
&\big|(Tu)(t)-(Tw)(t)\big| \\
&\le \Big|\big(I-\nu[1]\big)^{-1}\frac{1}{\Gamma(\alpha)}\Big|\Big\|\nu\Big(\int_0^t (t-s)^{\alpha-1}\Big(F\big(s,u(s)\big)-F\big(s,w(s)\big)\Big)ds\Big)\Big\| \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|F\big(s,u(s)\big)-F\big(s,w(s)\big)\big\|ds \\
&\le \frac{1}{\Gamma(\alpha)}\Big|\big(I-\nu[1]\big)^{-1}\|\nu\|+I\Big|\int_0^t (t-s)^{\alpha-1}B(s)\big\|u(s)-w(s)\big\|ds \\
&\le \frac{1}{\Gamma(\alpha)}\Big|\big(I-\nu[1]\big)^{-1}\|\nu\|+I\Big|\int_0^t (t-s)^{\alpha-1}B(s)ds\,\|u-w\|,
\end{aligned} \tag{3.10}$$

Since assumption (a) and Theorem 2.8, we get

$$\begin{aligned}
\big|(Tu)(t)-(Tw)(t)\big| &\le \frac{1}{\Gamma(\alpha)}\Big|\big(I-\nu[1]\big)^{-1}\|\nu\|+I\Big|\Big(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\Big)^{1-\alpha_1} \\
&\quad\times\Big(\int_0^t \big(B^{\frac{1}{\alpha_1}}(s)\big)ds\Big)^{\alpha_1}\|u-w\| \\
&\le \frac{1}{\Gamma(\alpha)}\Big|\big(I-\nu[1]\big)^{-1}\|\nu\|+I\Big|t^{\frac{\alpha-\alpha_1}{1-\alpha_1}}\|B\|_{L^{\frac{1}{\alpha_1}}[0,1]}\|u-w\| \\
&\le \frac{1}{\Gamma(\alpha)}\Big|\big(I-\nu[1]\big)^{-1}\|\nu\|+I\Big|\|B\|_{L^{\frac{1}{\alpha_1}}[0,1]}\|u-w\|.
\end{aligned} \tag{3.11}$$

Finally, (3.11) gives

$$\|Tu-Tw\| \le \frac{1}{\Gamma(\alpha)}\Big|\big(I-\nu[1]\big)^{-1}\|\nu\|+I\Big|\|B\|_{L^{\frac{1}{\alpha_1}}[0,1]}\|u-w\|. \tag{3.12}$$

Since by hypothesis, the spectral radius of the matrix $\frac{1}{\Gamma(\alpha)}|(I-\nu[1])^{-1}\|\nu\| + I|\|B\|_{L^{\frac{1}{\alpha_1}}[0,1]}$ is less than 1. Thus T is contractive and the conclusion follows from Perov's fixed point theorem.

We consider system (3.1) and the following inequations

$$\begin{cases} \left|D_0^\alpha \bar{u}(t) - F\big(t, \bar{u}(t)\big)\right| \leq \delta(t)\varepsilon \\ \bar{u}(0) \quad = \nu[\bar{u}], \end{cases} \tag{3.13}$$

where $\delta(t) : (0, 1) \to R_+$ and $\varepsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T, \quad \epsilon_i > 0, i = 1, 2, \ldots, n.$ □

Definition 3.2 System (3.1) is Hyers–Ulam stable with respect to System (3.13), if there exists $A_f > 0$, $A_f \in \mathsf{M}_{n\times n}(R_+)$ such that

$$\left\|\bar{u} - \tilde{u}\right\| \leq A_f \varepsilon,$$

for all $t \in J$ where $\bar{u}$ is any the solution of (3.13), and $\tilde{u}$ of the solution for system (3.1).

Theorem 3.3 *Assume $f_i : [0, 1] \times R^n \to R$, $(i = 1, 2, \ldots, n)$ are L^1-Carathéodory functions and satisfy the assumption* (a)*, $\nu_{ij} : C[0, 1] \to R, i, j = 1, 2, \cdots, n$ are linear and continuous functions and satisfy the assumption.* (b) *If the spectral radius of the matrix*

$$\frac{1}{\Gamma(\alpha)}\left|\big(I - \nu[1]\big)^{-1}\|\nu\| + I\right|\|B\|_{L^{\frac{1}{\alpha_1}}[0,1]}, \tag{3.14}$$

is less than 1, and in system (3.13) $\sup_{t\in(0,1)} \delta(t) \leq 1$. Then,the system (3.1) is Hyers–Ulam stable with respect to system (3.13).

Proof Let $h(t) = D_0^\alpha \bar{u}(t) - F(t, \bar{u}(t))$, consider the system

$$\begin{cases} D_0^\alpha \bar{u}(t) = F(t, \bar{u}(t)) + h(t) \\ \bar{u}(0) \quad = \nu[\bar{u}]. \end{cases} \tag{3.15}$$

Similarly to the system in Theorem 3.1, system (3.15) is equivalent to the following integral equation in $C^n[0, 1]$

$$\begin{aligned} \bar{u}(t) = &\big(I - \nu[1]\big)^{-1}\frac{1}{\Gamma(\alpha)}\nu\left[\int_0^t (t-s)^{\alpha-1}F\big(s, \bar{u}(s)\big)ds\right] \\ &+ \big(I - \nu[1]\big)^{-1}\frac{1}{\Gamma(\alpha)}\nu\left[\int_0^t (t-s)^{\alpha-1}h(s)ds\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F\big(s, \bar{u}(s)\big)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds. \end{aligned}$$

Now, we define the operator $\tilde{T}$ as following

$$\begin{aligned} (\tilde{T}\bar{u})(t) = &\big(I - \nu[1]\big)^{-1}\frac{1}{\Gamma(\alpha)}\nu\left[\int_0^t (t-s)^{\alpha-1}F\big(s, \bar{u}(s)\big)ds\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F\big(s, \bar{u}(s)\big)ds + H(t), \end{aligned} \tag{3.16}$$

where

$$H(t) = \left(I - \nu[1]\right)^{-1} \frac{1}{\Gamma(\alpha)} \nu\left[\int_0^t (t-s)^{\alpha-1} h(s)ds\right] + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds. \tag{3.17}$$

By consequence, we obtain

$$\|\tilde{T}\bar{u} - \tilde{T}\bar{w}\| = \|T\bar{u} - T\bar{w}\|.$$

Then the existence of a solution of (3.1) implies the existence of a solution to (3.15), it follows from Theorem 3.1 that $\tilde{T}$ is a contraction. So, there is a unique fixed point $\bar{u}$ of $\tilde{T}$, and respectively $\tilde{u}$ of T.

Since $t \in [0, 1]$ and $\sup_{t\in(0,1)} \delta(t) \leq 1$, we get

$$\begin{aligned}\|H\| &= \max_{t\in J} |H(t)| = \left|\left(I - \nu[1]\right)^{-1} \frac{1}{\Gamma(\alpha)} \nu\left[\int_0^t (t-s)^{\alpha-1} h(s)ds\right]\right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \right| \leq \frac{1}{\Gamma(\alpha+1)} \left|\left(I - \nu[1]\right)^{-1} \|\nu\| + I\right| \|h\| \\ &\leq \frac{1}{\Gamma(\alpha+1)} \left|\left(I - \nu[1]\right)^{-1} \|\nu\| + I\right| \varepsilon.\end{aligned}$$

Then we obtain

$$\begin{aligned}\|\bar{u} - \tilde{u}\| &= \|\tilde{T}(\bar{u}) - T\tilde{u}\| = \|T(\bar{u}) + H(t) - T(\tilde{u})\| \\ &\leq \|T(\bar{u}) - T(\tilde{u})\| + \|H(t)\| \\ &\leq \frac{1}{\Gamma(\alpha)} \left|\left(I - \nu[1]\right)^{-1} \|\nu\| + I\right| \|B\|_{L^{\frac{1}{\alpha_1}}[0,1]} \|\bar{u} - \tilde{u}\| \\ &\quad + \frac{1}{\Gamma(\alpha+1)} \left|\left(I - \nu[1]\right)^{-1} \|\nu\| + I\right| \varepsilon.\end{aligned}$$

Note that $I - \frac{1}{\Gamma(\alpha)}((I - \nu[1])^{-1} + I)\|B\|_{L^{\frac{1}{\alpha_1}}[0,1]}$ is invertible and its inverse is a nonnegative matrix since $\rho\left(\frac{1}{\Gamma(\alpha)}\left((I - \nu[1])^{-1} + I\right)\|B\|_{L^{\frac{1}{\alpha_1}}[0,1]}\right) < 1$, then we have

$$\begin{aligned}\|\bar{u} - \tilde{u}\| &\leq \left(I - \frac{1}{\Gamma(\alpha)} \left|\left(I - \nu[1]\right)^{-1} \|\nu\| + I\right| \|B\|_{L^{\frac{1}{\alpha_1}}[0,1]}\right)^{-1} \\ &\quad \frac{1}{\Gamma(\alpha+1)} \left|\left(I - \nu[1]\right)^{-1} \|\nu\| + I\right| \|h\| \\ &\leq \frac{1}{\Gamma(\alpha+1)} \left(I - \frac{1}{\Gamma(\alpha)} \left|\left(I - \nu[1]\right)^{-1} \|\nu\| + I\right| \|B\|_{L^{\frac{1}{\alpha_1}}[0,1]}\right)^{-1} \\ &\quad \left|\left(I - \nu[1]\right)^{-1} \|\nu\| + I\right| \varepsilon.\end{aligned} \tag{3.18}$$

Let $A_f = \frac{1}{\Gamma(\alpha+1)}\left(I - \frac{1}{\Gamma(\alpha)}\left|\left(I - \nu[1]\right)^{-1} + I\right|\left\|B\right\|_{L^{\frac{1}{\alpha_1}}[0,1]}\right)^{-1}\left|\left(I - \nu[1]\right)^{-1}\left\|\nu\right\| + I\right|$, and the condition $\sup\limits_{t\in(0,1)} \delta(t) \leq 1$, from (3.18) we obtain

$$\left\|\bar{u} - \tilde{u}\right\| \leq A_f \varepsilon.$$

This complete the proof. □

Example 3.4 Consider the nonlocal problem

$$\begin{cases} D_0^{\frac{1}{2}} u_1(t) = sin u_1 + u_2 + g_1(t), \\ D_0^{\frac{1}{2}} u_2(t) = cos(u_1 - u_2) + g_2(t), \\ u_1(0) = \frac{1}{4}\displaystyle\int_0^{\frac{1}{2}} \big(u_1(s) + u_2(s)\big) ds, \\ u_2(0) = \frac{1}{4}\displaystyle\int_0^{\frac{1}{2}} \big(u_1(s) + u_2(s)\big) ds, \end{cases}$$

where $t \in [0, 1]$, $g_i(t) \in L_1(0, 1)$, $i = 1, 2$. We have

$$\begin{cases} \nu[1] = \frac{1}{8}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \left(I - \nu[1]\right)^{-1} = \frac{1}{6}\begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix} \\ B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \left\|\nu\right\| = \frac{1}{8}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}. \end{cases} \tag{3.19}$$

Then

$$Q = \frac{1}{\Gamma(\alpha)}\left|\left(I - \nu[1]\right)^{-1}\left\|\nu\right\| + I\right|\left\|B\right\|_{L^{\frac{1}{\alpha_1}}[0,1]} = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and $\rho_1(Q) = 0 < 1$, $\rho_2(Q) = \frac{1}{\sqrt{\pi}} < 1$. From Theorem 3.1, the problem (3.19) has a unique solution u on $[0, 1]$. Furthermore, the solution u is Hyers–Ulam stable with respect to the following system

$$\begin{cases} \left|D_0^{\frac{1}{2}} \bar{u}_1(t) - sin\bar{u}_1 - \bar{u}_2 - g_1(t)\right| \leq \delta(t)\epsilon_1, \\ \left|D_0^{\frac{1}{2}} \bar{u}_2(t) - cos(\bar{u}_1 - \bar{u}_2) - g_2(t)\right| \leq \delta(t)\epsilon_2, \\ \bar{u}_1(0) = \frac{1}{4}\displaystyle\int_0^{\frac{1}{2}} \big(\bar{u}_1(s) + \bar{u}_2(s)\big) ds, \\ \bar{u}_2(0) = \frac{1}{4}\displaystyle\int_0^{\frac{1}{2}} \big(\bar{u}_1(s) + \bar{u}_2(s)\big) ds, \end{cases}$$

where $\epsilon_i > 0$, $i = 1, 2$, $\sup\limits_{t\in(0,1)} \delta(t) \leq 1$.

4 Existence and Hyers–Ulam stability results of system(1.3)–(1.4)

The problems (1.3)–(1.4) can be rewritten in the vector form

$$\begin{cases} D_0^{\alpha}(D+\lambda)u(t) = F\big(t, u(t)\big), \\ u(0) = 0 \\ u'(0) = \nu[u]. \end{cases} \tag{4.1}$$

where the functions u, F, ν are defined the same as in system(3.1).

We denote by $|u|_{C[a,b]}$ the usual max norm on $C[a,b]$, for each $\theta > 0$, by $|u|_{C_\theta[a,b]}$ the equivalent form

$$|u|_{C_\theta[a,b]} = \left|e^{\theta(t-a)}u(t)\right|_{C[a,b]} = \max_{t\in[a,b]} \big(e^{\theta(t-a)}u(t)\big), \tag{4.2}$$

we will use the vector-valued norms $|u|_{C_\theta[a,b]}$ in this section.

Let us make the following assumption

(c) the matrix $I - \lambda^{-1}\nu[1-e^{-\lambda t}]$ is nonsingular over $[0, 1]$.

Lemma 4.1 *If the assumption (c) is satisfied, then the system (4.1) is equivalent to the following integral equation*

$$\begin{aligned} u(t) = {} & \frac{\big(I - \lambda^{-1}\nu[1-e^{-\lambda t}]\big)^{-1}(1-e^{-\lambda t})}{\Gamma(\alpha)\lambda}\nu \\ & \times\left[\int_0^t e^{-\lambda(t-r)} \int_0^r (r-s)^{\alpha-1} F\big(s, u(s)\big)ds dr\right] \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-r)} \int_0^r (r-s)^{\alpha-1} F\big(s, u(s)\big)ds dr. \end{aligned} \tag{4.3}$$

Proof Applying the operator I^α on both sides of the first equation of system (4.1), we get

$$(D+\lambda)u(t) = C_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F\big(s, u(s)\big)ds,$$

which can be rewritten as

$$D\big(e^{\lambda t}u(t)\big) = C_0 e^{\lambda t} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F\big(s, u(s)\big)ds e^{\lambda t}.$$

Integrating from 0 to t,we get

$$\begin{aligned} u(t) = {} & \frac{C_0(1-e^{-\lambda t})}{\lambda} + C_1 e^{-\lambda t} \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-r)} \int_0^r (r-s)^{\alpha-1} F\big(s, u(s)\big)ds dr. \end{aligned}$$

Using the initial conditions in (4.1), we find $C_1 = 0$ and

$$\begin{aligned} C_0 &= u'(0) = \nu[u] \\ &= \nu\left[\frac{C_0(1-e^{-\lambda t})}{\lambda}\right] + \nu\left[\frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}F\big(s,u(s)\big)dsdr\right] \\ &= \frac{C_0}{\lambda}\nu\left[1-e^{-\lambda t}\right] + \nu\left[\frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}F\big(s,u(s)\big)dsdr\right]. \end{aligned} \tag{4.4}$$

Since the matrix $I - \lambda^{-1}\nu\left[1-e^{-\lambda t}\right]$ is nonsingular over [0, 1], we obtain

$$C_0 = \frac{1}{\Gamma(\alpha)}\Big(I - \lambda^{-1}\nu\left[1-e^{-\lambda t}\right]\Big)^{-1}\nu\left[\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}F\big(s,u(s)\big)dsdr\right]. \tag{4.5}$$

Substituting these values of $C_1 = 0$, and C_0 in (4.5), we get (4.3). This completes the proof. □

Theorem 4.2 *Assume $f_i : [0,1]\times R^n \to R$, $(i = 1, 2, \dots, n)$ are L^1-Carathéodory functions and satisfy the assumption (a) with $B(t) \in L_2([0,1]; \mathrm{M}_{n\times n})$, $\nu_{ij} : C[0,1] \to R, i, j = 1, 2, \dots, n$ are linear and continuous functions and satisfy the assumption (c). If the spectral radius of the matrix*

$$\frac{\sqrt{e^{2\lambda}-1}}{\sqrt{2\lambda}\,\Gamma(\alpha+1)}\left|I + \left(\lambda^{-1}I - \nu\left[1-e^{-\lambda t}\right]\right)^{-1}\|\nu\|\right|_{t\in[0,1]}\|B\|_{L_2[0,1]}, \tag{4.6}$$

is less than 1, then the problem (4.1) has a unique solution in X.

Proof Define the operator $\phi : C([0,1], X) \to C([0,1], X)$ by

$$\begin{aligned} (\phi u)(t) = {} & \frac{\Big(I - \lambda^{-1}\nu\left[1-e^{-\lambda t}\right]\Big)^{-1}(1-e^{-\lambda t})}{\Gamma(\alpha)\lambda}\nu \\ & \times\left[\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}F\big(s,u(s)\big)dsdr\right] \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}F\big(s,u(s)\big)dsdr. \end{aligned} \tag{4.7}$$

we want to proof the operator ϕu has a fixed point.

We recall that (see [20])

$$\int_s^t e^{\lambda r}(r-s)^{\alpha-1}dr = (-1)^{\alpha-1}\lambda^{-\alpha}e^{\lambda s}\Gamma\big(\alpha, -\lambda(t-s), 0\big),$$

where

$$\Gamma\big(\alpha, -\lambda(t-u), 0\big) = \Gamma\big(\alpha, -\lambda(t-u)\big) - \Gamma(\alpha, 0) = \int_{-\lambda(t-u)}^{0} r^{\alpha-1}e^{-r}dr.$$

Hence we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}F\big(s,u(s)\big)dsdr\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t F\big(s,u(s)\big)\int_s^t e^{-\lambda(t-r)}(r-s)^{\alpha-1}drds\\
&= \frac{(-1)^{\alpha-1}\lambda^{-\alpha}e^{-\lambda t}}{\Gamma(\alpha)}\int_0^t F\big(s,u(s)\big)e^{\lambda s}\Gamma\big(\alpha,-\lambda(t-s),0\big)ds.
\end{aligned}\tag{4.8}$$

For any function $u, w \in C\big([0, 1], X\big)$, and $t \in [0, 1]$, we have

$$\begin{aligned}
\big|(\phi u)(t) - (\phi w)(t)\big| &\le \frac{\lambda^{-\alpha}}{\Gamma(\alpha)}\big|I + \big(\lambda^{-1}I - \nu[1-e^{-\lambda t}]\big)^{-1}\|\nu\|\big|_{t\in[0,1]}\\
&\quad \int_0^t e^{\lambda s}\Gamma\big(\alpha,-\lambda(t-s),0\big)\big\|F\big(s,u(s)\big) - F\big(s,w(s)\big)\big\|ds\\
&\le \frac{\lambda^{-\alpha}}{\Gamma(\alpha)}\big|I + \big(\lambda^{-1}I - \nu[1-e^{-\lambda t}]\big)^{-1}\|\nu\|\big|_{t\in[0,1]}\\
&\quad \int_0^t \Gamma(\alpha,-\lambda(t-s),0)B(s)\big\|e^{\lambda s}\big(u(s)-w(s)\big)\big\|ds\\
&\le \frac{\lambda^{-\alpha}}{\Gamma(\alpha)}\big|I + \big(\lambda^{-1}I - \nu[1-e^{-\lambda t}]\big)^{-1}\|\nu\|\big|_{t\in[0,1]}\\
&\quad \int_0^t \Gamma\big(\alpha,-\lambda(t-s),0\big)B(s)ds\big\|(u-w)\big\|_{C_{\lambda[0,1]}}\\
&\le \frac{\lambda^{-\alpha}}{\Gamma(\alpha)}\big|I + \big(\lambda^{-1}I - \nu[1-e^{-\lambda t}]\big)^{-1}\|\nu\|\big|_{t\in[0,1]}\\
&\quad \left(\int_0^t \Gamma^2\big(\alpha,-\lambda(t-s),0\big)ds\right)^{\frac{1}{2}}\left(\int_0^t B^2(s)ds\right)^{\frac{1}{2}}\\
&\quad \times\big\|(u-w)\big\|_{C_{\lambda[0,1]}},
\end{aligned}\tag{4.9}$$

where $\|(u-w)\|_{C_{\lambda[0,1]}}$ is the vector norm defined as (4.2). Since

$$\begin{aligned}
\Gamma^2\big(\alpha,-\lambda(t-s),0\big) &= \left(\int_{-\lambda(t-s)}^{0} r^{\alpha-1}e^{-r}dr\right)^2\\
&= \left(\int_0^{\lambda(t-s)} r^{\alpha-1}e^{r}dr\right)^2
\end{aligned}$$

$$\leq \left(e^{\lambda(t-s)} \int_0^{\lambda(t-s)} r^{\alpha-1} dr \right)^2$$
$$= e^{2\lambda(t-s)} \left(\frac{1}{\alpha} \lambda^\alpha (t-s)^\alpha \right)^2. \tag{4.10}$$

From (4.9) and (4.10), we obtain that

$$\big|(\phi u)(t) - (\phi w)(t)\big| \leq \frac{\sqrt{e^{2\lambda}-1}}{\sqrt{2\lambda}\Gamma(\alpha+1)} \big|I + \big(\lambda^{-1}I - \nu[1-e^{-\lambda t}]\big)^{-1} \big\|\nu\big\|\big|_{[0,1]} \big\|B\big\|_{L_2[0,1]} \big\|(u-w)\big\|_{C_{\lambda[0,1]}}. \tag{4.11}$$

Therefore,the Perov fixed point theorem implies that coupled system (4.1) has a unique solution in X. And we complete the proof. □

Definition 4.3 The system (4.1) is called Hyers–Ulam stable with respect to the following system

$$\begin{cases} |D_0^\alpha (D+\lambda)\hat{u}(t) - F\big(t, \hat{u}(t)\big)| \leq \tilde{\delta}(t)\varepsilon, \\ \hat{u}(0) = 0, \\ \hat{u}^{'}(0) = \nu[\hat{u}], \end{cases} \tag{4.12}$$

here $\delta(t) : (0,1) \to R_+$ and $\varepsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T, \epsilon_i > 0, (i = 1, 2, \ldots, n)$, if there exist a constant matrix $\tilde{A}_f \in M_{n\times n}(R_+)$ such that for the solution u of system (4.1) and each solution $\hat{u}$ of system (4.12) satisfy

$$\big\|u - \hat{u}\big\| \leq \tilde{A}_f \varepsilon.$$

Theorem 4.4 *Assume $f_i : [0,1] \times R^n \to R, (i = 1, 2, \ldots, n)$ are L^1-Carathéodory functions and satisfy the assumption (a) with $B(t) \in L_2\big([0,1]; \mathrm{M}_{n\times n}\big)$, $\nu_{ij} : C[0,1] \to R, i, j = 1, 2, \ldots, n$ are linear and continuous functions and satisfy the assumption (c). If the spectral radius of the matrix*

$$\frac{\sqrt{e^{2\lambda}-1}}{\sqrt{2\lambda}\Gamma(\alpha+1)} \big|I + \Big(\lambda^{-1}I - \nu\big[1-e^{-\lambda t}\big]\Big)^{-1} \big\|\nu\big\|\big|_{t\in[0,1]} \big\|B\big\|_{L_2[0,1]}, \tag{4.13}$$

is less than 1, and $\sup_{t\in(0,1)} \delta(t) \leq 1$, then the system (4.1) is Ulam–Hyers stable with respect to system (4.12).

Proof Let the operator $\tilde{\phi}$ as

$$(\tilde{\phi}\hat{u})(t) = \frac{\left(I - \lambda^{-1}\nu\left[1 - e^{-\lambda t}\right]\right)^{-1}\left(1 - e^{-\lambda t}\right)}{\Gamma(\alpha)\lambda}\nu\Bigg[\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1} \times F\big(s, \hat{u}(s)\big)dsdr + \int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}\tilde{h}(s)dsdr\Bigg] + \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}F\big(s, \hat{u}(s)\big)dsdr + \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}\tilde{h}(s)dsdr, \tag{4.14}$$

where

$$\tilde{h}(t) = D_0^{\alpha}(D+\lambda)\hat{u}(t) - F\big(t, \hat{u}(t)\big), \quad t \in [0, 1]. \tag{4.15}$$

If we denote

$$\tilde{H}(t) = \frac{\left(I - \lambda^{-1}\nu\left[1 - e^{-\lambda t}\right]\right)^{-1}(1 - e^{-\lambda t})}{\Gamma(\alpha)\lambda}\nu\left[\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}\tilde{h}(s)dsdr\right] + \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-r)}\int_0^r (r-s)^{\alpha-1}\tilde{h}(s)dsdr. \tag{4.16}$$

Using the with the condition $\rho\left(\frac{\sqrt{e^{2\lambda}-1}}{\sqrt{2\lambda}\Gamma(\alpha+1)}\left|I + \left(\lambda^{-1}I - \nu\left[1-e^{-\lambda t}\right]\right)^{-1}\right\|\nu\|\big|_{t\in[0,1]} \|B\|_{L_2[0,1]}\right) < 1$, and Theorem 2.6 (c), with similar thinking in Theorem 3.3, we obtain

$$\begin{aligned}\|\hat{u} - u\| &\leq \left(I - \frac{\sqrt{e^{2\lambda}-1}}{\sqrt{2\lambda}\Gamma(\alpha+1)}\left|I + \left(\lambda^{-1}I - \nu\left[1 - e^{-\lambda t}\right]\right)^{-1}\right\|\nu\|\big|_{[0,1]}\|B\|_{L_2[0,1]}\right)^{-1} \\ &\quad \frac{\left(I - \lambda^{-1}\nu\left[1 - e^{-\lambda t}\right]\right]^{-1}\sqrt{e^{2\lambda}-1}}{\Gamma(\alpha+1)\sqrt{2\lambda}}\|\tilde{h}\|_{\lambda[0,1]} \\ &\leq \left(I - \frac{\sqrt{e^{2\lambda}-1}}{\sqrt{2\lambda}\Gamma(\alpha+1)}\left|I + \left(\lambda^{-1}I - \nu\left[1-e^{-\lambda t}\right]\right)^{-1}\right\|\nu\|\big|_{[0,1]}\|B\|_{L_2[0,1]}\right)^{-1} \\ &\quad \frac{\left(I - \lambda^{-1}\nu\left[1 - e^{-\lambda t}\right]\right]^{-1}\sqrt{e^{2\lambda}-1}}{\Gamma(\alpha+1)\sqrt{2\lambda}}\varepsilon. \end{aligned} \tag{4.17}$$

Denote

$$\tilde{A}_f = \left(I - \frac{\sqrt{e^{2\lambda}-1}}{\sqrt{2\lambda}\Gamma(\alpha+1)}\left|I + (\lambda^{-1}I - \nu\left[1 - e^{-\lambda t}\right])^{-1}\right\|\nu\|\big|_{[0,1]}\|B\|_{L_2[0,1]}\right)^{-1} \frac{\left(I - \lambda^{-1}\nu\left[1 - e^{-\lambda t}\right]\right]^{-1}\sqrt{e^{2\lambda}-1}}{\Gamma(\alpha+1)\sqrt{2\lambda}}.$$

Then we have

$$\left\|\hat{u}-u\right\| \leq \tilde{A}_f \varepsilon.$$

By the Definition 4.3, the solution u of (4.1) is Hyers–Ulam stable with respect to system (4.12). The proof is completed. □

Example 4.5 Consider the nonlocal problem

$$\begin{cases} D_0^{\frac{1}{2}}(D+1)u_1(t) = \frac{1}{12}sinu_1 + \frac{1}{12}tan^{-1}u_2 + g_1(t), \\ D_0^{\frac{1}{2}}(D+1)u_2(t) = \frac{1}{12}cos(u_1-u_2) + g_2(t), \\ u_1(0)=0, u_2(0)=0, \\ u_1^{'}(0) = u_1(\frac{1}{2}) - u_2(\frac{1}{2}), \\ u_2^{'}(0) = u_1(\frac{1}{2}) + u_2(\frac{1}{2}), \end{cases} \tag{4.18}$$

where $t \in [0,1]$, $g_i(t) \in L_1(0,1)$. We have

$$\begin{cases} \nu\big[1-e^{-\frac{1}{2}}\big] = \big(1-e^{-\frac{1}{2}}\big)\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \\ \Big(I-\nu\big[1-e^{-\frac{1}{2}}\big]\Big)^{-1} = \frac{1}{2e^{-1}-2e^{-\frac{1}{2}}+1}\begin{pmatrix} e^{-\frac{1}{2}} & 1-e^{-\frac{1}{2}} \\ e^{-\frac{1}{2}}-1 & e^{-\frac{1}{2}} \end{pmatrix}, \\ B = \frac{1}{12}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ \|\nu\| = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \Gamma\big(1+\frac{1}{2}\big) = \frac{1}{2}\sqrt{\pi}. \end{cases}$$

Then

$$\begin{aligned} \tilde{Q} &= \frac{\sqrt{e^{2\lambda}-1}}{\sqrt{2\lambda}\Gamma(\alpha+1)}\left| I + \Big(\lambda^{-1}I - \nu\big[1-e^{-\lambda t}\big]\Big)^{-1}\|\nu\|\right|_{t\in[0,1]} \|B\|_{L_2[0,1]} \\ &= \frac{\sqrt{2(e^2-1)}}{12\sqrt{\pi}(2e^{-1}-2e^{-\frac{1}{2}}+1)}\begin{pmatrix} 2e^{-1}-2e^{-\frac{1}{2}}+3 & 2e^{-1}-2e^{-\frac{1}{2}}+3 \\ 2e^{-\frac{1}{2}}+2e^{-1}-1 & 2e^{-\frac{1}{2}}+2e^{-1}-1 \end{pmatrix}, \end{aligned}$$

and $\rho_1(Q) = 0 < 1$, $\rho_2(Q) = \frac{(2+4e^{-1})(\sqrt{2(e^2-1)})}{12\sqrt{\pi}(2e^{-1}-2e^{-\frac{1}{2}}+1)} < 1$. From the result of Theorem 4.2, the problem (4.18) has a unique solution u on $[0,1]$. Furthermore, by Theorem 4.4, the solution u is Hyers–Ulam stable with respect to the following system

$$\begin{cases} |D_0^{\frac{1}{2}}(D+1)\hat{u}_1(t) - \frac{1}{12}sin\hat{u}_1 - \frac{1}{12}tan^{-1}\hat{u}_2 - g_1(t)| \leq \delta(t)\epsilon_1, \\ |D_0^{\frac{1}{2}}(D+1)\hat{u}_2(t) - \frac{1}{12}cos(\hat{u}_1-\hat{u}_2) - g_2(t)| \leq \delta(t)\epsilon_2, \\ \hat{u}_1(0)=0, \hat{u}_2(0)=0, \\ \hat{u}_1(0) = \hat{u}_1(\frac{1}{2}) + \hat{u}_2(\frac{1}{2}), \\ \hat{u}_2(0) = \hat{u}_1(\frac{1}{2}) - \hat{u}_2(\frac{1}{2}), \end{cases}$$

where $\epsilon_i > 0, \quad i = 1, 2, \; \delta(t) : (0, 1) \to R, \; \sup\limits_{t \in (0,1)} \delta(t) \le 1.$

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